

## On unicoherence at subcontinua

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## ON UNICOHERENCE AT SUBCONTINUA\*

By

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**Abstract.** In this paper an Eilenberg-type characterization of unicoherence at subcontinua and a mapping property about this unicoherence are given.

In [5], a localization of the notion of unicoherence, i.e., unicoherence at subcontinua was introduced. Several mapping properties about unicoherence at subcontinua are studied in [1]. This property is related to other properties of unicoherence closely.

The main purpose of this paper is to establish an Eilenberg-type characterization of unicoherence at continua and to show that local homeomorphism preserves unicoherence at subcontinua for locally connected continua. The latter partially answers a question raised by J.J. Charatonik in [1].

### 1. Preliminary

A continuum is a compact connected metric space. A continuum is unicoherent if the intersection of every two subcontinua having union  $X$  is connected; a continuum  $X$  is hereditarily unicoherent if every subcontinuum of  $X$  is unicoherent. Let  $Y$  be a subcontinuum of  $X$ ;  $X$  is unicoherent at  $Y$ , denoted  $U_n(Y)$ , if for each pair of proper subcontinua  $A$  and  $B$  of  $X$  such that  $X = A \cup B$  the set  $A \cap B \cap Y$  is connected.

Let  $S^1$  denote the unit circle. The mapping  $f \in S^{1X}$  is said to be inessential ( $f \sim 1$ ) if there exists a mapping  $\phi \in R^X$  such that  $f(x) = e^{i\phi(x)}$  for every  $x \in X$ .

The mapping  $f \in S^{1X}$  is said to be inessential on the subspace  $Y$  of  $X$  ( $f \sim 1$  on  $Y$ ), if there exists a mapping  $\phi \in R^Y$  such that  $f(x) = e^{i\phi(x)}$  for every  $x \in Y$ .

S. Eilenberg introduced the property (b) for studying unicoherence. A continuum  $X$  is said to have property (b) if for each mapping  $f \in S^{1X}$ , there is

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$f \sim 1$  ([3] p.63).

We say that a continuum  $X$  has property (b) on a subcontinuum  $Y$  of  $X$  if for each mapping  $f \in S^{1^X}$ , there is  $f \sim 1$  on  $Y$ .

It is clear to have

LEMMA 1. *Suppose that a continuum  $X$  has property (b) on a subcontinuum  $Y$  of  $X$  and  $Z$  is a subcontinuum of  $Y$ . Then  $X$  has property (b) on  $Z$  and  $Y$  also has property (b) on  $Z$ .*

PROPOSITION 2 ([3]). *Any continuum which has property (b) is unicoherent.*

PROPOSITION 3 ([3]). *Let continuum  $X$  be locally connected. The following conditions are equivalent:*

- (1)  $X$  is unicoherent;
- (2)  $X$  has property (b).

PROPOSITION 4 ([5] corollary 1.5). *Let  $Y_1, Y_2, \dots, Y_n$  be a finite collection of subcontinua of a continuum  $X$  such that  $X$  is  $U_n(Y_i)$  for every  $i$ , and suppose that for each  $i > 1$*

$$Y_i \cap \bigcup \{Y_j : j < i\} \neq \emptyset.$$

*Then  $X$  is  $U_n(\bigcup_{i=1}^n Y_i)$ .*

PROPOSITION 5 ([5] Theorem 1.6). *Let  $Y$  be a subcontinuum of a continuum  $X$ . If  $X$  is  $U_n(Y)$  and  $A$  and  $B$  are proper subcontinua of  $X$  such that  $X = A \cup B$ , then the sets  $A \cap Y$  and  $B \cap Y$  are connected.*

PROPOSITION 6. *If  $Y$  is a subcontinuum of a continuum  $X$  and  $Z$  is a subcontinuum of  $Y$ . Suppose that  $X$  is  $U_n(Y)$  and  $Y$  is  $U_n(Z)$ . Then  $X$  is  $U_n(Z)$ .*

PROOF. Assuming that the conclusion is false, then there is a pair of proper subcontinua  $A$  and  $B$  of  $X$  such that  $X = A \cup B$  and  $A \cap B \cap Z$  is not connected. Suppose  $A \cap B \cap Z = H \cup K$  is a separation. Because  $X$  is  $U_n(Y)$ , by proposition 5,  $A \cap Y$  and  $B \cap Y$  are all connected. One can assume that both of  $A \cap Y$  and  $B \cap Y$  are a nonempty proper subcontinuum of  $Y$  (Otherwise the case is simple). Then  $[(A \cap Y) \cap (B \cap Y)] \cap Z = (A \cap B \cap Y) \cap Z = (A \cap B \cap Z) \cap Y = (H \cup K) \cap Y = H \cup K$ . Since  $H \cup K \subset Z \subset Y$ , this contradicts to that  $Y$  is  $U_n(Z)$ .

PROPOSITION 7 (Corollary 7 of [1]). *Monotone mappings preserve uni-*

*coherence at subcontinua.*

## 2. An Eilenberg-type characterization of unicoherence at subcontinua

In this section we give a characterization of unicoherence at subcontinua which is similar with the characterization of unicoherence given by S. Eilenberg.

**THEOREM 8.** *Suppose that  $Y$  is a subcontinuum of a continuum  $X$  and for each pair of proper subcontinua  $A$  and  $B$  of  $X$  has property (b) on  $A \cap B \cap Y$ . Then  $X$  is unicoherent at  $Y$ .*

**PROOF.** Suppose  $X$  does not be unicoherent at  $Y$ . Then there are subcontinua  $A$  and  $B$  of  $X$  such that  $X = A \cup B$  and  $A \cap B \cap Y$  is not connected. Write  $A \cap B \cap Y$  as a disjoint union of nonempty closed subsets  $C$  and  $D$ . One can assume that  $A \cap Y \neq \emptyset \neq B \cap Y$ . Otherwise  $A \cap B \cap Y$  must be connected. Define a function  $\phi: X \rightarrow \mathbb{R}$  by

$$\phi(x) = \pi \frac{d(x, C)}{d(x, C) + d(x, D)},$$

for each  $x \in X$ , and mapping  $f: X \rightarrow S^1$  by

$$f(x) = \begin{cases} e^{i\phi(x)}, & \text{if } x \in A, \\ e^{-i\phi(x)}, & \text{if } x \in B. \end{cases}$$

Thus the mapping  $f$  is well defined and continuous. By hypothesis of property (b) on  $A \cap B \cap Y$ , one have that  $f \sim 1$  on  $A \cap B \cap Y$ . Then there is a  $\xi \in R^{A \cap B \cap Y}$  such that  $f(x) = e^{i\xi(x)}$  for each  $x \in A \cap B \cap Y$ . According to Proposition 5 both of  $A \cap Y$  and  $B \cap Y$  are connected. There exist integers  $m$  and  $n$  such that

$$\phi(x) = \xi(x) + 2m\pi, \quad \text{if } x \in A \cap Y$$

and

$$-\phi(x) = \xi(x) + 2n\pi, \quad \text{if } x \in B \cap Y.$$

However, if  $x \in C \subset A \cap B \cap Y$ , then  $\xi(x) = -2m\pi = -2n\pi$  and hence  $m=n$ . On the other hand, if  $x \in D \subset A \cap B \cap Y$ , then  $\xi(x) = \pi - 2m\pi = -\pi - 2n\pi$  and get  $\pi = -\pi$ . This contradiction establishes the Theorem.

**THEOREM 9.** *Let  $X$  be a locally connected continuum and  $Y$  is its subcontinuum. The following conditions are equivalent:*

- (1)  $X$  is unicoherent at  $Y$ ;

(2)  $X$  has property (b) on  $Y$ .

PROOF. Theorem 7 has established  $(2) \Rightarrow (1)$ . We prove  $(1) \Rightarrow (2)$ . For every  $f: X \rightarrow S^1$  one can let  $f = f_2 \circ f_1$  by the monotone-light factorization of  $f$ , where  $f_1 X \rightarrow X'$  is monotone mapping and  $f_2: X' \rightarrow S^1$  is light mapping. By Proposition 7,  $X'$  is unicoherent at  $Y' = f_1(Y)$ . Since  $f_2$  is light,  $X'$  must be at most 1-dimensional. Hence  $Y'$  must contain no any simple closed curve since  $X'$  is  $U_n(Y)$ .  $Y'$  is a locally connected continuum that contains no simple closed curve, i.e., it is a dendrite. Thus  $Y'$  is unicoherent and locally connected continuum. By Eilenberg's characterization,  $Y'$  has property (b). It is not difficult to see that  $X'$  has property (b) on  $Y'$ . This means that there is a  $\psi \in S^{1Y'}$  such that  $f_2(x') = e^{i\psi(x')}$ , for each  $x' \in Y'$ . Let  $\phi = \psi \circ f_1 \in S^{1Y}$ . Then  $f(x) = f_2 f_1(x) = e^{i\phi(x)}$ , for each  $x \in Y$ . This is  $f \sim 1$  on  $Y$ .

### 3. Unicoherence at continua under local homeomorphism

It is known that a surjective mapping on a continuum is a local homeomorphism if and only if it is open and  $n$ -to-1 for some fixed  $n \geq 1$ . It is proved that open finite-to-one mapping do not preserve unicoherence at subcontinua, even if the domain space is a linear graph in [1]. In the paper J.J. Charatonik raised a question: Do local homeomorphism perserve unicoherence at continua?

THEOREM 10. Suppose  $X$  be a locally connected continuum,  $Y$  is a subcontinuum of  $X$  and  $X$  is  $U_n(Y)$ ,  $f: X \rightarrow X'$  is a local homeomorphism. Then  $X'$  is  $U_n(Y')$ , here  $Y' = f(Y)$ .

PROOF. Whole proof consists of three steps.

CLAIM 1.  $Y$  can be covered by finite subcontinua  $Y_1, Y_2, \dots, Y_m$  such that  $X$  is  $U_n(Y_i)$  and  $f|_{Y_i}$  is a homeomorphis, for  $i = 1, \dots, m$ .

Since  $X$  is locally connected, for any  $x \in Y$  there is a connected open neighborhood  $V_x$  of  $x$  in  $Y$ . Moreover one can assume that  $f|_{\bar{V}_x}$  a homeomorphism because  $f$  is a local homeomorphism.

The local connectedness of  $X$  and its unicoherence at  $Y$  imply property (b) on  $Y$  by Theorem 9, i.e., for each  $f: X \rightarrow S^1 f|_Y \sim 1$ . Thus, by Proposition 1,  $f|_{\bar{V}_x} \sim 1$  and this means that  $X$  is  $U_n(\bar{V}_x)$ . By compactness of  $Y$ , finite subcontinua  $Y_1, \dots, Y_m$  as required above can be found. Denote  $Y'_i = f(Y_i)$ .

CLAIM 2.  $X'$  is  $U_n(Y'_i)$ .

For any pair of proper subcontinua  $A'$  and  $B'$  of  $X'$  such that  $A' \cup B' = X'$  we'll show that  $A' \cap B' \cap Y'_i$  is connected. Local homeomorphism between continua is exactly a  $n$ -to-one open continuous mapping ([1]) and it is a confluent mapping ([4]). Thus one can get disjoint unions of subcontinua of  $X$

$$f^{-1}(A') = A_1 \cup \dots \cup A_k \quad \text{and} \quad f^{-1}(B') = B_1 \cup \dots \cup B_s,$$

here  $k, s \leq n$  and each of  $A_j$  and  $B_j$  is mapped onto  $A'$  and  $B'$  by  $f$  respectively. Since  $f|_{Y_i}$  is a homeomorphism, it is not difficult to see that only one of  $A_j$  and  $B_j$  intersects  $Y_i$  respectively. Assume that  $A_1$  and  $B_1$  are they. One can consider two subcontinua

$$A_1 \cup \{B_j : B_j \cap A_1 \neq \emptyset\} \quad \text{and} \quad B_1 \cup \{A_j : A_j \cap B_1 \neq \emptyset\}.$$

Similarly, consider the rest of  $A_j$  and  $B_j$  which meet  $A_1 \cup \{B_j : B_j \cap A_1 \neq \emptyset\}$  or  $B_1 \cup \{A_j : A_j \cap B_1 \neq \emptyset\}$  and continue to form unions. Because numbers of  $A_j$  and  $B_j$  are finite, so finally get two continua

$$A = A_1 \cup \{B_j : j \in J' \subset \{2, \dots, s\}\} \cup \{A_j : j \in J \subset \{2, \dots, k\}\}$$

and

$$B = B_1 \cup \{A_j : j \in \{2, \dots, k\} \setminus J\} \cup \{B_j : j \in \{2, \dots, s\} \setminus J'\}.$$

Then  $A \cup B = X$  and  $A_1 \cap B_1 \cap Y_i = A \cap B \cap Y_i$  is connected by unicoherence at  $Y_i$ . Therefore  $A' \cap B' \cap Y'_i = f(A_1 \cap B_1 \cap Y_i)$  is also connected.

CLAIM 3. It is from Claim 2 that  $X'$  is  $U_n(Y')$ .

$Y' = f(Y) = \bigcup_{i=1}^m Y'_i$ . The finite collection  $\{Y'_1, \dots, Y'_m\}$  can be selected such that for each  $i > 1$ ,  $Y'_i \cap \bigcup\{Y'_j : j < i\} \neq \emptyset$ . By Corollary 1.5 of [5], the final conclusion yields.

A unicoherent continuum  $X$  is strong unicoherent if for every pair of proper subcontinua  $A$  and  $B$  such that  $X = A \cup B$  both  $A$  and  $B$  are unicoherent. By Theorem 10 and Theorem 2.1 of [5] we have.

COROLLARY 11. *An image of a locally connected strongly unicoherent continuum under a local homeomorphism is locally connected strongly unicoherent.*

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